Cheat sheet – λ-calculus

BNF grammar

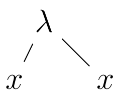
$$
\begin{aligned}
T ::= \; & x && (\textit{variable})\\
\mid \; & (TT) && (\textit{application})\\
\mid \; & (\lambda x.T) && (\lambda\textit{-abstraction})
\end{aligned}
$$

Parenthesis

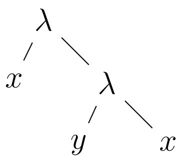
- application is left-associative: $T_1T_2T_3 = T_1(T_2T_3)$
- λ-abstraction is right-associative: $\lambda x.\lambda y.T = \lambda x.(\lambda y.T)$ and $\lambda x.T_1T_2 = \lambda x.(T_1T_2)$
- $(T_1T_2) = T_1T_2$

Well-known terms

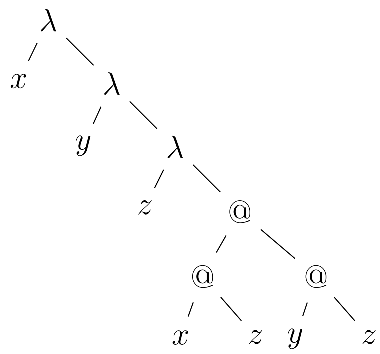
- $\mathrm{I} = \lambda x.x$



- $\mathrm{K} = \lambda x.\lambda y.x$



- $\mathrm{S} = \lambda x.\lambda y.\lambda z.((xz)(yz))$



Free variables *vs* bound variables

Free variable	bound variable
defined outside a term	intern to the term
name is essential (cannot be modified)	name is not important (can be modified)

Inductive definition of Free Variables FV: $\begin{cases} FV(x) = \{x\} \\ FV(T_1T_2) = FV(T_1) \cup FV(T_2) \\ FV(\lambda x.T) = FV(T) \backslash \{x\} \end{cases}$

Substitution

$[x \mapsto T_1]T_2$ is the term defined by replacing all free occurrences of x within T_2 by T_1

Inductive definition of substitution on $\Lambda_{\mathcal{X}}$:

(1) $[x \mapsto T]x = T$

(2) $[x \mapsto T]y = y$ if $x \neq y$

(3) $[x \mapsto T]T_1T_2 = [x \mapsto T]T_1[x \mapsto T]T_2$

(4) $[x \mapsto T]\lambda y.T' = \lambda y.[x \mapsto T]T'$ if $x \neq y$ and $y \notin FV(T)$

α-conversion or α-equivalence (*renaming*)

renaming a defining occurrence and all its depending bound occurrences

$$\lambda x.T =_\alpha \lambda y.[x \mapsto y]T \quad \text{if } y \notin FV(T)$$

e.g.: $\lambda x.x =_\alpha \lambda y.y$ but $\lambda x.y \neq_\alpha \lambda x.z$

β-reduction

$\lambda x.T_1T_2 \to [x \mapsto T_2]T_1$ can be applied anywhere in a term