IMT Atlantique
Bretagne-Pays de la Loire
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PC2 – Let's practice induction

Langages et logique – ELU 610

Objectives

At the end of the activity, you should be capable of:

- defining an object by induction;
- make simple proof by induction.

Exercise 1 (*Subterms*)

▷ **Question 1.1:**
Is there an occurrence of the term $T_1 = yx(xz)$ in the term $T_2 = zyx(xz)x$?

▷ **Question 1.2:**
Is there an occurrence of the term $(\lambda x.\lambda y.\lambda z.(xz)(yz))u$ in the following terms:

$$\begin{cases} T_1 = (\lambda x.\lambda y.\lambda z.xz(yz))uvw \\ T_2 = w(\lambda x.\lambda y.\lambda z.xz(yz))uv \end{cases}$$

▷ **Question 1.3:**
Define the function sub which computes the set of subterms of a term.

Exercise 2 (*Binary Trees*)

A is a set values. A binary tree over A is a either empty or contains a value from A and has a left child and a right child.

▷ **Question 2.1:**
Define inductively the set $\mathcal{B}_A$ of binary trees over A.

▷ **Question 2.2:**
Define the function $|\cdot|$ which compute the number of nodes of a binary tree.

▷ **Question 2.3:**
Define the function h which compute the height of a binary tree. The height is defined the longest path between the root of the tree and a leaf.

▷ **Question 2.4:**
Prove that for any binary tree T, $|T| \leq 2^{h(T)+1} - 1$

Exercise 3 (*Church Integers*)

Let $\underline{0}$ be the combinator defined by $\lambda s.\lambda z.z$. Let it represents zero. Let $\underline{\mathsf{succ}}$ be the function $\lambda v.\lambda s.\lambda z.s\,(vsz)$.

▷ **Question 3.1:**
If we use $\underline{\mathsf{succ}}$ as the usual successor function of integers what is the encoding of $\underline{n}$ for $n \in \mathbb{N}$? Prove your proposition by induction.

▷ **Question 3.2:**
Prove that usual addition, multiplication and exponentiation can be defined by:

$$\begin{cases} \underline{+} = \lambda nm.(n\,\underline{\mathsf{succ}}\,m) \\ \underline{\times} = \lambda nm.(n\,(\underline{+}\,m)\,\underline{0}) \end{cases}$$

▷ **Question 3.3:**
Propose a term for exponentiation.