IMT Atlantique
Bretagne-Pays de la Loire
École Mines-Télécom

The λ-calculus

Mathematical modeling of functions

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ELU 610 – C5

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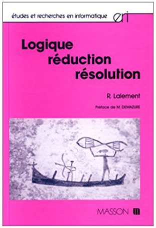
The λ-calculus [Church]

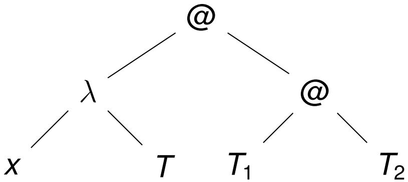
- A formal language proposed by Alonzo Church in the 1930s to model the notion of function
- http://en.wikipedia.org/wiki/Lambda_calculus
- We will use it to
  - illustrate the notion of formal language
  - understand fundamentals of formal reasoning
  - introduce the functional paradigm

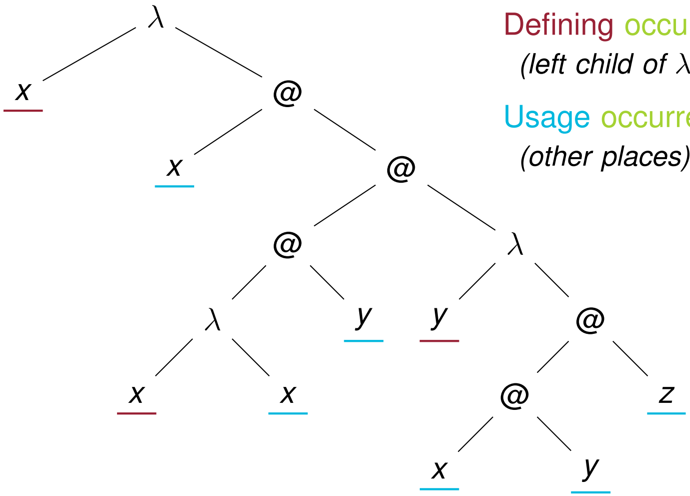


René Lalement
Logique Réduction Résolution
ERI Masson, 1990
Book translated in english *Computation as logic*, Prentice-Hall in 1993, ISBN 9780137700097

- The set $\Lambda_{\mathcal{X}}$ of the terms defined by
  - variables x, y, ... from a denumerable set $\mathcal{X}$
  - applications $(T_1 T_2)$ of a term T_1 (the function) to a term T_2 (the argument)
  - functions $(\lambda x.T)$ of a variable x (the parameter) and a term T (the body)
- BNF: $T ::= x \mid (TT) \mid (\lambda x.T)$
- Parenthesis may be omitted
  - outer: $(T_1 T_2) = T_1 T_2$ and $(\lambda x.T) = \lambda x.T$
  - application is left associative: $T_1 T_2 T_3 = (T_1 T_2) T_3$
  - λ is right associative: $\lambda x.\lambda y.T = \lambda x.(\lambda y.T)$ and $\lambda x.T_1 T_2 = \lambda x.(T_1 T_2)$
- Some well-known λ-terms
  - $\lambda x.x = \mathbf{I}$ $\lambda x.\lambda y.x = \mathbf{K}$ $\lambda x.\lambda y.\lambda z.((xz)(yz)) = \mathbf{S}$

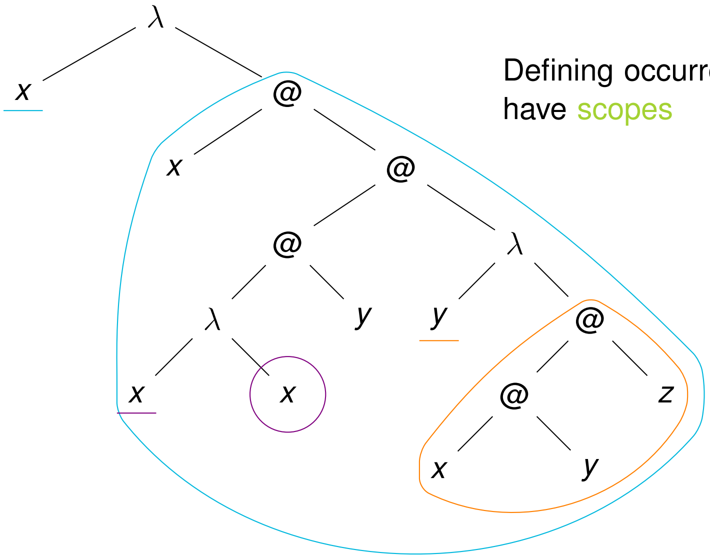
- $\Lambda_{\mathcal{X}} = T_{\{@,\lambda\}}[\mathcal{X}]$ with
  - @ is the only constructor and $ar(@) = 2$
  - λ is the only binder and $ar(\lambda) = 1$
- Terms are trees
  - variables are leaves
  - constructors and binders are nodes
- ex: $(\lambda x.T)(T_1 T_2)$



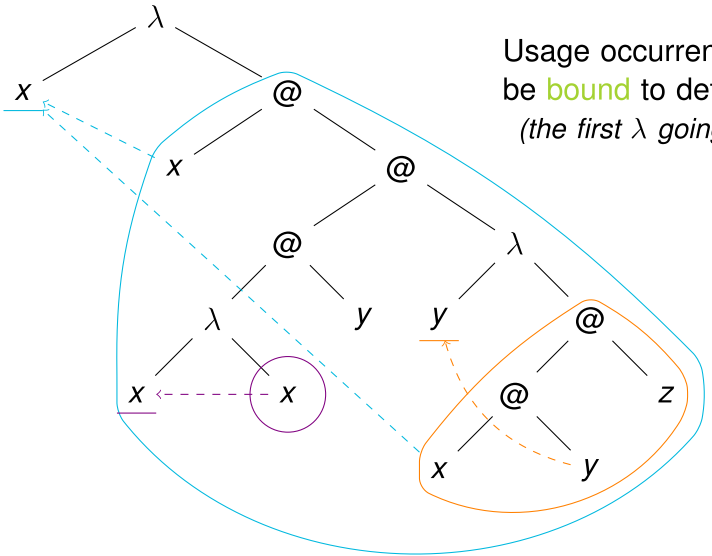


Defining occurrences
(left child of λ)

Usage occurrences
(other places)

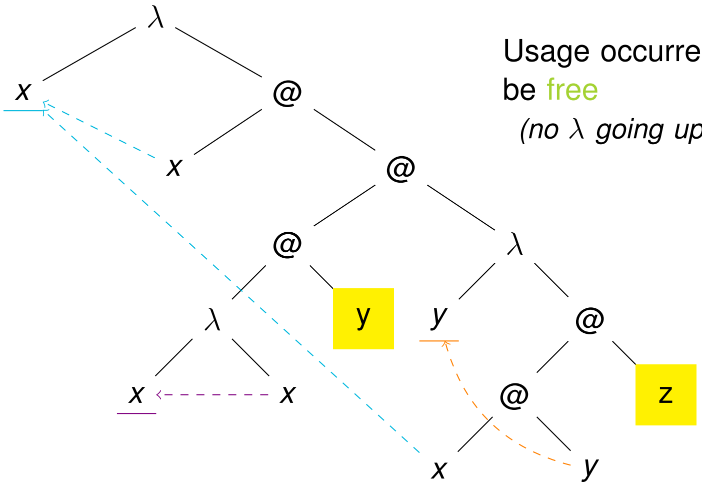


Defining occurrences have scopes



Usage occurrences may be bound to defining ones

(the first λ going up)



Usage occurrences may be free

(no λ going up)

- A free variable is defined outside the term
  - a kind of *global variable* (for the term)
  - its name is essential and cannot be modified
  - $\lambda x.y$ is different from $\lambda x.z$
- A bound variable is intern to the term
  - a kind of *local variable* (for the term)
  - its name can be modified (the defining occurrence and all its depending bound occurrences)
  - $\lambda x.x$ is identical to $\lambda y.y$
  - known as α-conversion (see later for the mathematical definition)
  - the name of a bound variable has no importance, only the link to its binder[1]
- A term with free variables is open
- A term with no free variables is closed (*a.k.a.* combinators)

[1] there exists notations without names, see for example [Bou08]

- One can define a function f on $\mathbb{N}$ recursively by
  1. defining $f(0)$
  2. defining $f(n+1)$ in terms of $f(n)$

  for example, factorial
  1. $0! = 1$
  2. $(n+1)! = (n+1)n!$
- One can prove a property P on $\mathbb{N}$ by
  1. proving $P(0)$
  2. proving that if $P(n)$ holds, $P(n+1)$ is true

  for example, if $P(n)$ is $0 + 1 + \cdots + n = \frac{n(n+1)}{2}$
  1. $0 = 0$
  2. $0 + 1 + \cdots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = (n+1)(\frac{n}{2} + 1)$
     $= \frac{(n+1)(n+2)}{2}$
- Variants: starting at k or $P(0), \ldots, P(n) \Rightarrow P(n+1)$

- (Structural) induction is a method of definition or proof on the set of terms $T_\Sigma[\mathcal{X}]$
- One can define a function f on $\Lambda_\mathcal{X}$ inductively by
  1. defining f on $\mathcal{X}$ (leaves)
  2. defining $f(T_1 T_2)$ in terms of $f(T_1)$ and $f(T_2)$
  3. defining $f(\lambda x.T)$ in terms of $f(T)$
- For example, the set of free variables FV is defined by
  1. $FV(x) = \{x\}$
  2. $FV(T_1 T_2) = FV(T_1) \cup FV(T_2)$
  3. $FV(\lambda x.T) = FV(T) \setminus \{x\}$
- For example, the size of a λ-term is defined by
  1. $\mathtt{size}(x) = 1$
  2. $\mathtt{size}(T_1 T_2) = \mathtt{size}(T_1) + \mathtt{size}(T_2) + 1$
  3. $\mathtt{size}(\lambda x.T) = \mathtt{size}(T) + 1$

- One can prove a property P on $\Lambda_{\mathcal{X}}$ inductively by
  1. proving P on $\mathcal{X}$
  2. proving $P(T_1 T_2)$ supposing $P(T_1)$ and $P(T_2)$ are true
  3. proving $P(\lambda x.T)$ supposing $P(T)$ is true
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$$
\begin{aligned}
card(FV(T_1\,T_2)) &= card(FV(T_1) \cup FV(T_2)) && \text{def of } FV \\
&\leq card(FV(T_1)) + card(FV(T_2)) && \text{prop of } card \\
&\leq \texttt{size}(T_1) + \texttt{size}(T_2) && \text{IH} \\
&\leq \texttt{size}(T_1\,T_2) && \text{def of } \texttt{size}
\end{aligned}
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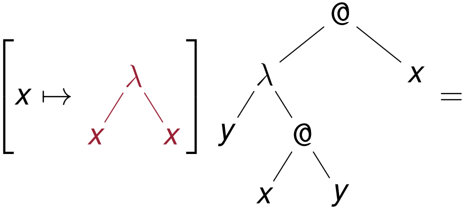
  3. let's suppose $\mathit{card}(FV(T)) \leq \mathtt{size}(T)$ (IH)

$$\begin{aligned} \mathit{card}(FV(\lambda x.T)) &= \mathit{card}(FV(T) \setminus \{x\}) && \text{def of } FV \\ &\leq \mathit{card}(FV(T)) && \text{prop of } \mathit{card} \\ &\leq \mathtt{size}(T) && \text{IH} \\ &\leq \mathtt{size}(\lambda x.T) && \text{def of } \mathtt{size} \end{aligned}$$

- Giving a meaning to a free variable is done by substitution
- Substitution is a function associating a term to
  - a variable (the substituted variable) and
  - two terms (the replacement term and the term on which substitution operates)
- $[x \mapsto T_1]\,T_2$ is the term defined by replacing **all** free occurrences of x within T_2 by T_1



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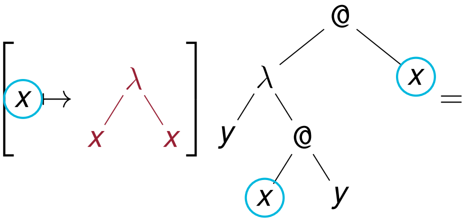

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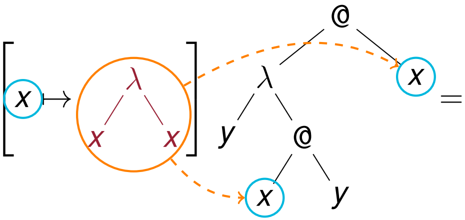

Substitutions

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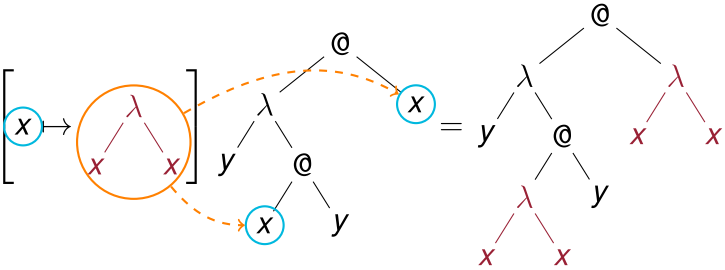

- Defined inductively

$$\begin{cases} [x \mapsto T]x &= T & \\ [x \mapsto T]y &= y & \text{if } x \neq y \\ [x \mapsto T]T_1 T_2 &= [x \mapsto T]T_1 [x \mapsto T]T_2 & \\ [x \mapsto T]\lambda y.T' &= \lambda y.[x \mapsto T]T' & \text{if } x \neq y, y \notin FV(T) \end{cases}$$

  the last condition prevent captures of a free y in T
- The definition is incomplete *e.g.* $[x \mapsto T]\lambda x.T'$, $[x \mapsto y]\lambda y.T$
- α-conversion (a.k.a. α-equivalence) is defined by
  $\lambda x.T =_\alpha \lambda y.[x \mapsto y]T \quad$ if $y \notin FV(T)$ (*freshness condition*)
- The definition of substitution is complete modulo renaming
  - if $x = y$ or $y \in FV(M)$, we rename the bound y
- We always work on $\Lambda_{\mathcal{X}}/=_\alpha$ (modulo renaming)

$$\begin{cases} (1)\ [x \mapsto T]x = T & \\ (2)\ [x \mapsto T]y = y & \text{if } x \neq y \\ (3)\ [x \mapsto T]T_1 T_2 = [x \mapsto T]T_1 [x \mapsto T]T_2 & \\ (4)\ [x \mapsto T]\lambda y.T' = \lambda y.[x \mapsto T]T' & \text{if } x \neq y \text{ and } y \notin FV(T) \\ & \\ (\alpha)\ \lambda x.T = \lambda y.[x \mapsto y]T & \text{if } y \notin FV(T) \end{cases}$$

- $[z \mapsto \lambda x.xy]\lambda z.x\ (\lambda y.zy) =$

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(3) $= [z \mapsto \lambda x.xy]\lambda z.x\ ([z \mapsto \lambda x.xy]\lambda y.zy)$

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$$\begin{aligned} &(3) && = [z \mapsto \lambda x.xy]\lambda z.x\ ([z \mapsto \lambda x.xy]\lambda y.zy) \\ &(\alpha)(\alpha) && = [z \mapsto \lambda x.xy](\lambda t.[z \mapsto t]x)\ ([z \mapsto \lambda x.xy](\lambda u.[y \mapsto u]zy)) \end{aligned}$$

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$$\begin{aligned} &\text{(3)} && = [z \mapsto \lambda x.xy]\lambda z.x\ ([z \mapsto \lambda x.xy]\lambda y.zy) \\ &(\alpha)(\alpha) && = [z \mapsto \lambda x.xy](\lambda t.[z \mapsto t]x)\ ([z \mapsto \lambda x.xy](\lambda u.[y \mapsto u]zy)) \\ &\text{(2)(3,2+1)} && = [z \mapsto \lambda x.xy]\lambda t.x\ ([z \mapsto \lambda x.xy]\lambda u.zu) \end{aligned}$$

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- Everyone should be comfortable with such rewritings…

- The usual function call can be modeled by

$$\underbrace{(\lambda x.T_1)}_{(1)}\underbrace{T_2}_{(2)} \to \underbrace{[x \mapsto T_2]T_1}_{(3)}$$

  where (1) is the function, (2) the argument and (3) the result
- For example $\mathbf{II} = \lambda x.x\ \lambda x.x \to [x \mapsto \lambda x.x]x = \lambda x.x = \mathbf{I}$
- This rule is called β-reduction (def later)
- It can be applied anywhere within a term
- A location in a term where it can be applied is called a β-redex

- A judgment is a logical assertion, here[2]: $Term \rightarrow OtherTerm$
- An inference rule is a set of judgments $J_1, \ldots, J_n, J$ such that $J_1 \wedge \ldots \wedge J_n \Rightarrow J$
  - $J_1, \ldots, J_n$ are the premises, J is the conclusion
  - written $$\frac{J_1 \quad \cdots \quad J_n}{J}$$
  - An axiom is an inference rule with no premise
- A derivation is a tree of such rules where the leaves are axioms

$$\dfrac{\dfrac{}{J_1} \quad \dfrac{}{J_2} \quad \cdots \quad \dfrac{\dfrac{}{J_3} \quad \cdots \quad \dfrac{}{J_4}}{J_5}}{J_6}$$

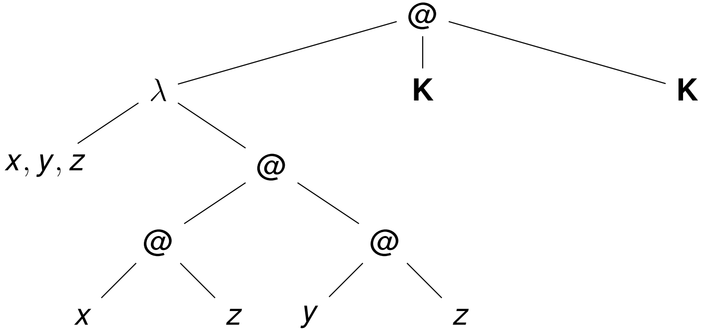
see http://en.wikipedia.org/wiki/Inference_rule

[2]There exists various other forms of judgment

$$(1)\ (\lambda x.T_1)T_2 \rightarrow [x \mapsto T_2]T_1 \qquad (2)\ \frac{T \rightarrow T'}{\lambda x.T \rightarrow \lambda x.T'}$$

$$(3)\ \frac{T_1 \rightarrow T_1'}{T_1 T_2 \rightarrow T_1' T_2} \qquad (4)\ \frac{T_2 \rightarrow T_2'}{T_1 T_2 \rightarrow T_1 T_2'}$$

$$\begin{aligned}
\mathbf{SKK} &= \underline{\lambda x}.\lambda y.\lambda z.((xz)(yz))\underline{\mathbf{K}}\mathbf{K} && \text{def of } \mathbf{S} \\
&\rightarrow \underline{\lambda y}.\lambda z.((\mathbf{K}z)(yz))\underline{\mathbf{K}} && (1) \\
&\rightarrow \lambda z.((\mathbf{K}z)(\mathbf{K}z)) && (1) \\
&\rightarrow \lambda z.(((\underline{\lambda x}.\lambda y.x)\underline{z})(\mathbf{K}z)) && \text{def of } \mathbf{K} \\
&\rightarrow \lambda z.(\underline{\lambda y}.z\underline{(\mathbf{K}z)}) && (1) \\
&\rightarrow \lambda z.z && (1) \\
&\rightarrow \mathbf{I} && \text{def of } \mathbf{I}
\end{aligned}$$


@
λ
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x, y, z
@
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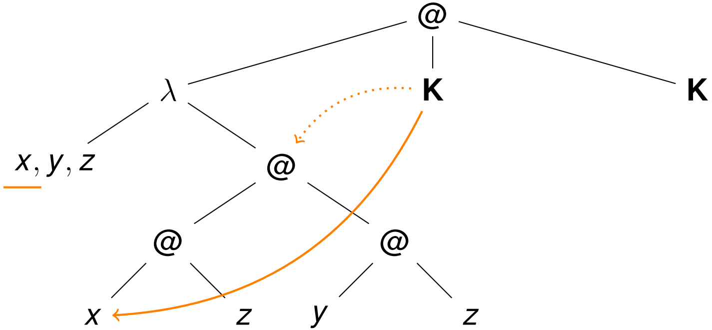
@
λ
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x, y, z
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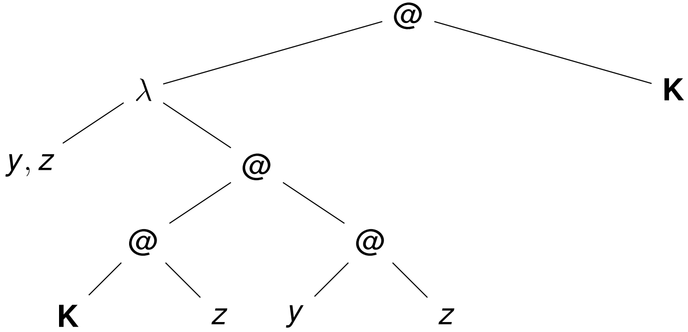
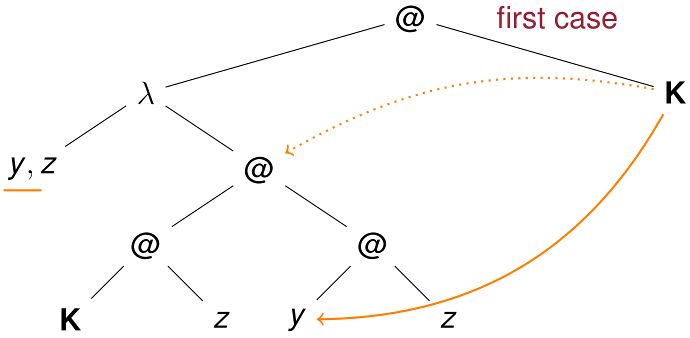
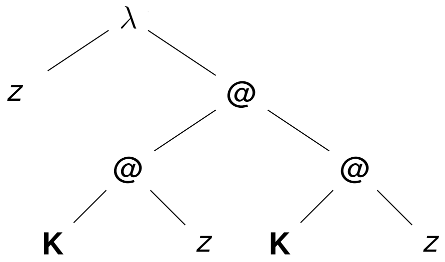
x
z
y
z


@
λ
K
y, z
@
@
@
K
z
y
z


@
first case
λ
K
y, z
@
@
@
K
z
y
z

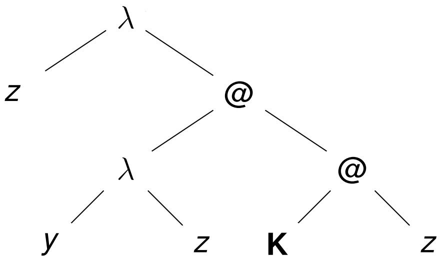
Another look

first case



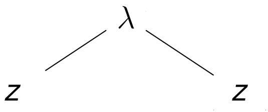
- $\mathbf{K}z = (\lambda x.\lambda y.x)z = \lambda y.z$

first case

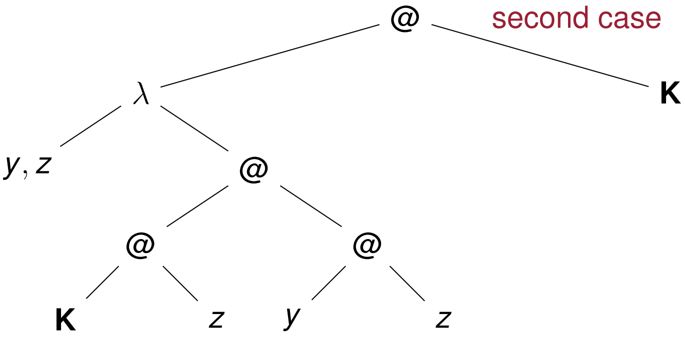


- $\mathbf{K}z = (\lambda x.\lambda y.x)z = \lambda y.z$
- $(\lambda y.z)T = z$

first case



- $\mathbf{K}z = (\lambda x.\lambda y.x)z = \lambda y.z$
- $(\lambda y.z)T = z$



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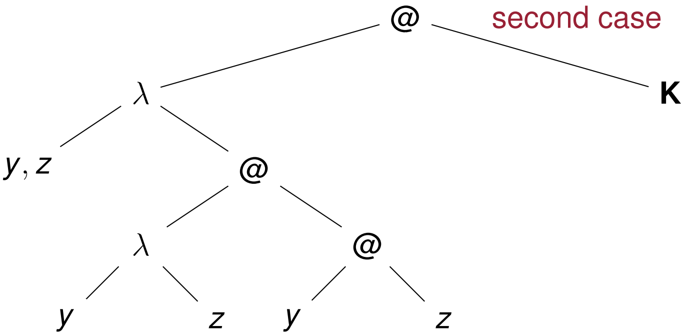

- $\mathbf{K}z = (\lambda x.\lambda y.x)z = \lambda y.z$
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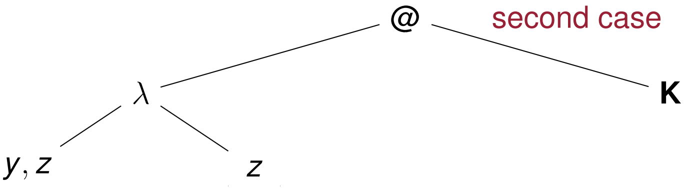
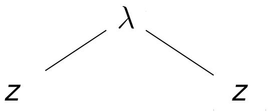


- $\mathbf{K}z = (\lambda x.\lambda y.x)z = \lambda y.z$
- $(\lambda y.z)T = z$

second case



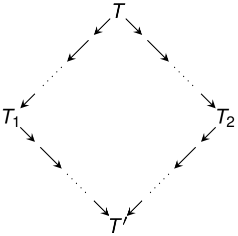
same result!

- $\mathbf{K}z = (\lambda x.\lambda y.x)z = \lambda y.z$
- $(\lambda y.z)T = z$

- A term T is irreducible or normal, if there exist no term it can reduce to ($T \not\to$)
- If T reduces to T' normal, T' is called a normal form of T
- A reduction sequence is a sequence $T_1 \to \cdots \to T_n$
  - denoted $T_1 \to^n T_n$
  - denoted $T_1 \to^* T_n$ if you don't care about the number of steps
- Often, there is several reduction sequences starting from a term (*e.g.* **SKK**)
- A reduction (resp. a term) is
  - (strongly) normalizing if all (resp. its) reduction sequences are finite
  - weakly normalizing if all terms have (resp. it has) a normal form
- $\Omega = (\lambda x.xx)(\lambda x.xx) \to \Omega$

⚠ β-reduction is not weakly normalizing for $\Lambda_{\mathcal{X}}$

- If T reduces to T_1 and T_2 there exists T' such that T_1 and T_2 both reduce to T'



- It shows that the path of computation is not important
- A term has at most one normal form

Church-Rosser theorem

β-reduction is confluent on $\Lambda_{\mathcal{X}}$

⚠ Some terms reduces indefinitely but has a normal form:
$\mathbf{KI}\Omega \to \mathbf{KI}\Omega$ or $\mathbf{KI}\Omega \to^2 \mathbf{I}$

Reduction strategy

- A reduction strategy is a way to choose the β-redex to reduce
- Standard orders
  - Normal order
    - the leftmost outermost reduction
    - always finds the normal form if it exists
  - Applicative order
    - the leftmost innermost reduction
    - only finds the normal form for normalizing terms
  - but both reduce inside functions (rule (2))
- Two other classical strategies (not using rule (2))
  - call by name: resolve application before evaluating the arguments
    - may duplicate computations
  - call by value: evaluate argument before application
    - optimal for sharing of computations

- In theory, yes as everything can be encoded as a λ
  - Turing has proved all computable functions can be written in $\Lambda_{\mathcal{X}}$
- In practice not usable, what is this term[3]?
  - $\lambda xyzu.(x(yzu)u)\lambda xy.(y(yx))\lambda xy.(yx)$

- We extend its core with
  - basic datatypes (integer, boolean, ...)
  - data structures (pairs, lists, ...)
  - recursion
  - ...

  It's the functional core of Ocaml! http://caml.inria.fr/ocaml

[3] We use $\lambda xyz.$ for $\lambda x.\lambda y.\lambda z.$, this notation si called currying

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  - . . .

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Conclusion

- The λ-calculus
  - anything that is computable can be expressed
  - is often used to study sequential computation
  - close to a programing language (Caml)
  - for the interested [Lal90]
- Used to illustrate fundamental notions
  - variables, scope
  - induction
  - substitution
  - reduction
- Starting point to learn functional programming

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Théorie des ensembles.
Eléments de mathématique. Springer, 2008.

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